

# Hamilton Dynamics on Clifford Kähler Manifolds

Mehmet Tekkoyun \*

Department of Mathematics, Pamukkale University,

20070 Denizli, Turkey

February 24, 2009

## Abstract

This paper presents Hamilton dynamics on Clifford Kähler manifolds. In the end, the some results related to Clifford Kähler dynamical systems are also discussed.

**Keywords:** Clifford Kähler Geometry, Hamiltonian Dynamics.

**PACS:** 02.40.

---

\*Corresponding author. E-mail address: tekkoyun@pau.edu.tr; Tel: +902582953616; Fax: +902582953593

# 1 Introduction

Modern differential geometry explains explicitly the dynamics of Hamiltons. So, if  $Q$  is an  $m$ -dimensional configuration manifold and  $\mathbf{H} : T^*Q \rightarrow \mathbf{R}$  is a regular Hamilton function, then there is a unique vector field  $X$  on  $T^*Q$  such that dynamic equations are determined by

$$i_X \Phi = d\mathbf{H} \quad (1)$$

where  $\Phi$  indicates the symplectic form. The triple  $(T^*Q, \Phi, X)$  is called *Hamilton system* on the cotangent bundle  $T^*Q$ .

At last time, there are many studies and books about Hamilton mechanics, formalisms, systems and equations such that real, complex, paracomplex and other analogues [1, 2] and there in. Therefore it is possible to obtain different analogous in different spaces.

It is known that quaternions were invented by Sir William Rowan Hamilton as an extension to the complex numbers. Hamilton's defining relation is most succinctly written as:

$$i^2 = j^2 = k^2 = ijk = -1 \quad (2)$$

If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. A lot of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basic principles in terms of quaternion algebra. It is well-known that quaternions are useful for representing rotations in both quantum and classical mechanics [3] . It is well known that Clifford manifold is a quaternion manifold. So, all properties defined on quaternion manifold of dimension  $8n$  also

is valid for Clifford manifold. Hence, it may be constructed mechanical equations on Clifford Kähler manifold.

The paper is structured as follows. In second 2, we review Clifford Kähler manifolds. In second 3 we introduce Hamilton equations for dynamical systems on Clifford Kähler manifold. In conclusion, we discuss some geometric-physical results about Hamilton equations and fields constructed on the base manifold.

## 2 Preliminaries

In this paper, all mappings and manifolds are assumed to be smooth, i.e. infinitely differentiable and the sum is taken over repeated indices. By  $\mathcal{F}(M)$ ,  $\chi(M)$  and  $\Lambda^1(M)$  we understand the set of functions on  $M$ , the set of vector fields on  $M$  and the set of 1-forms on  $M$ , respectively.

### 2.1 Clifford Kähler Manifolds

Here, we recall and extend the main concepts and structures given in [4, 5, 6] . Let  $M$  be a real smooth manifold of dimension  $m$ . Suppose that there is a 6-dimensional vector bundle  $V$  consisting of  $F_i (i = \overline{1,6})$  tensors of type (1,1) over  $M$ . Such a local basis  $\{F_1, F_2, \dots, F_6\}$  is named a canonical local basis of the bundle  $V$  in a neighborhood  $U$  of  $M$ . Then  $V$  is called an almost Clifford structure in  $M$ . The pair  $(M, V)$  is named an almost Clifford manifold with  $V$ . Thus, an almost Clifford manifold  $M$  is of dimension  $m = 8n$ . If there exists on  $(M, V)$  a global basis  $\{F_1, F_2, \dots, F_6\}$ , then  $(M, V)$  is called an almost Clifford manifold; the basis  $\{F_1, F_2, \dots, F_6\}$  is said to be a global basis for  $V$ .

An almost Clifford connection on the almost Clifford manifold  $(M, V)$  is a linear connection

$\nabla$  on  $M$  which preserves by parallel transport the vector bundle  $V$ . This means that if  $\Phi$  is a cross-section (local-global) of the bundle  $V$ , then  $\nabla_X \Phi$  is also a cross-section (local-global, respectively) of  $V$ ,  $X$  being an arbitrary vector field of  $M$ .

If for any canonical basis  $\{J_i\}$ ,  $i = \overline{1, 6}$  of  $V$  in a coordinate neighborhood  $U$ , the identities

$$g(J_i X, J_i Y) = g(X, Y), \quad \forall X, Y \in \chi(M), \quad i = 1, 2, \dots, 6, \quad (3)$$

hold, the triple  $(M, g, V)$  is called an almost Clifford Hermitian manifold or metric Clifford manifold denoting by  $V$  an almost Clifford structure  $V$  and by  $g$  a Riemannian metric and by  $(g, V)$  an almost Clifford metric structure.

Since each  $J_i$  ( $i = 1, 2, \dots, 6$ ) is almost Hermitian structure with respect to  $g$ , setting

$$\Phi_i(X, Y) = g(J_i X, Y), \quad i = 1, 2, \dots, 6, \quad (4)$$

for any vector fields  $X$  and  $Y$ , we see that  $\Phi_i$  are 6 local 2-forms.

If the Levi-Civita connection  $\nabla = \nabla^g$  on  $(M, g, V)$  preserves the vector bundle  $V$  by parallel transport, then  $(M, g, V)$  is named a Clifford Kähler manifold, and an almost Clifford structure  $\Phi_i$  of  $M$  is said to be a Clifford Kähler structure. Suppose that let

$$\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}, x_{4n+i}, x_{5n+i}, x_{6n+i}, x_{7n+i}\}, i = \overline{1, n}$$

be a real coordinate system on  $(M, V)$ . Then we denote by

$$\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}}, \frac{\partial}{\partial x_{4n+i}}, \frac{\partial}{\partial x_{5n+i}}, \frac{\partial}{\partial x_{6n+i}}, \frac{\partial}{\partial x_{7n+i}} \right\}, \quad (5)$$

$$\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}, dx_{4n+i}, dx_{5n+i}, dx_{6n+i}, dx_{7n+i}\}$$

the natural bases over  $\mathbf{R}$  of the tangent space  $T(M)$  and the cotangent space  $T^*(M)$  of  $M$ ,

respectively. By structures  $\{J_1, J_2, J_3, J_4, J_5, J_6\}$  the following expressions are given

$$\begin{aligned}
J_1\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{n+i}} & J_2\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{2n+i}} & J_3\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{3n+i}} \\
J_1\left(\frac{\partial}{\partial x_{n+i}}\right) &= -\frac{\partial}{\partial x_i} & J_2\left(\frac{\partial}{\partial x_{n+i}}\right) &= -\frac{\partial}{\partial x_{4n+i}} & J_3\left(\frac{\partial}{\partial x_{n+i}}\right) &= -\frac{\partial}{\partial x_{5n+i}} \\
J_1\left(\frac{\partial}{\partial x_{2n+i}}\right) &= \frac{\partial}{\partial x_{4n+i}} & J_2\left(\frac{\partial}{\partial x_{2n+i}}\right) &= -\frac{\partial}{\partial x_i} & J_3\left(\frac{\partial}{\partial x_{2n+i}}\right) &= -\frac{\partial}{\partial x_{6n+i}} \\
J_1\left(\frac{\partial}{\partial x_{3n+i}}\right) &= \frac{\partial}{\partial x_{5n+i}} & J_2\left(\frac{\partial}{\partial x_{3n+i}}\right) &= \frac{\partial}{\partial x_{6n+i}} & J_3\left(\frac{\partial}{\partial x_{3n+i}}\right) &= -\frac{\partial}{\partial x_i} \\
J_1\left(\frac{\partial}{\partial x_{4n+i}}\right) &= -\frac{\partial}{\partial x_{2n+i}} & J_2\left(\frac{\partial}{\partial x_{4n+i}}\right) &= \frac{\partial}{\partial x_{n+i}} & J_3\left(\frac{\partial}{\partial x_{4n+i}}\right) &= \frac{\partial}{\partial x_{7n+i}} \\
J_1\left(\frac{\partial}{\partial x_{5n+i}}\right) &= -\frac{\partial}{\partial x_{3n+i}} & J_2\left(\frac{\partial}{\partial x_{5n+i}}\right) &= -\frac{\partial}{\partial x_{7n+i}} & J_3\left(\frac{\partial}{\partial x_{5n+i}}\right) &= \frac{\partial}{\partial x_{n+i}} \\
J_1\left(\frac{\partial}{\partial x_{6n+i}}\right) &= \frac{\partial}{\partial x_{7n+i}} & J_2\left(\frac{\partial}{\partial x_{6n+i}}\right) &= -\frac{\partial}{\partial x_{3n+i}} & J_3\left(\frac{\partial}{\partial x_{6n+i}}\right) &= \frac{\partial}{\partial x_{2n+i}} \\
J_1\left(\frac{\partial}{\partial x_{7n+i}}\right) &= -\frac{\partial}{\partial x_{6n+i}} & J_2\left(\frac{\partial}{\partial x_{7n+i}}\right) &= \frac{\partial}{\partial x_{5n+i}} & J_3\left(\frac{\partial}{\partial x_{7n+i}}\right) &= -\frac{\partial}{\partial x_{4n+i}} \\
J_4\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{4n+i}} & J_5\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{5n+i}} & J_6\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{6n+i}} \\
J_4\left(\frac{\partial}{\partial x_{n+i}}\right) &= -\frac{\partial}{\partial x_{2n+i}} & J_5\left(\frac{\partial}{\partial x_{n+i}}\right) &= -\frac{\partial}{\partial x_{3n+i}} & J_6\left(\frac{\partial}{\partial x_{n+i}}\right) &= -\frac{\partial}{\partial x_{7n+i}} \\
J_4\left(\frac{\partial}{\partial x_{2n+i}}\right) &= \frac{\partial}{\partial x_{n+i}} & J_5\left(\frac{\partial}{\partial x_{2n+i}}\right) &= -\frac{\partial}{\partial x_{7n+i}} & J_6\left(\frac{\partial}{\partial x_{2n+i}}\right) &= -\frac{\partial}{\partial x_{3n+i}} \\
J_4\left(\frac{\partial}{\partial x_{3n+i}}\right) &= -\frac{\partial}{\partial x_{7n+i}} & J_5\left(\frac{\partial}{\partial x_{3n+i}}\right) &= \frac{\partial}{\partial x_{n+i}} & J_6\left(\frac{\partial}{\partial x_{3n+i}}\right) &= \frac{\partial}{\partial x_{2n+i}} \\
J_4\left(\frac{\partial}{\partial x_{4n+i}}\right) &= -\frac{\partial}{\partial x_i} & J_5\left(\frac{\partial}{\partial x_{4n+i}}\right) &= \frac{\partial}{\partial x_{6n+i}} & J_6\left(\frac{\partial}{\partial x_{4n+i}}\right) &= \frac{\partial}{\partial x_{5n+i}} \\
J_4\left(\frac{\partial}{\partial x_{5n+i}}\right) &= \frac{\partial}{\partial x_{6n+i}} & J_5\left(\frac{\partial}{\partial x_{5n+i}}\right) &= -\frac{\partial}{\partial x_i} & J_6\left(\frac{\partial}{\partial x_{5n+i}}\right) &= -\frac{\partial}{\partial x_{4n+i}} \\
J_4\left(\frac{\partial}{\partial x_{6n+i}}\right) &= -\frac{\partial}{\partial x_{5n+i}} & J_5\left(\frac{\partial}{\partial x_{6n+i}}\right) &= -\frac{\partial}{\partial x_{4n+i}} & J_6\left(\frac{\partial}{\partial x_{6n+i}}\right) &= -\frac{\partial}{\partial x_i} \\
J_4\left(\frac{\partial}{\partial x_{7n+i}}\right) &= \frac{\partial}{\partial x_{3n+i}} & J_5\left(\frac{\partial}{\partial x_{7n+i}}\right) &= \frac{\partial}{\partial x_{2n+i}} & J_6\left(\frac{\partial}{\partial x_{7n+i}}\right) &= \frac{\partial}{\partial x_{n+i}}.
\end{aligned} \tag{6}$$

A canonical local basis  $\{J_1^*, J_2^*, J_3^*, J_4^*, J_5^*, J_6^*\}$  of  $V^*$  of the cotangent space  $T^*(M)$  of manifold

$M$  satisfies the following condition:

$$J_1^{*2} = J_2^{*2} = J_3^{*2} = J_4^{*2} = J_5^{*2} = J_6^{*2} = -I, \tag{7}$$

being

$$\begin{aligned}
J_1^*(dx_i) &= dx_{n+i} & J_2^*(dx_i) &= dx_{2n+i} & J_3^*(dx_i) &= dx_{3n+i} \\
J_1^*(dx_{n+i}) &= -dx_i & J_2^*(dx_{n+i}) &= -dx_{4n+i} & J_3^*(dx_{n+i}) &= -dx_{5n+i} \\
J_1^*(dx_{2n+i}) &= dx_{4n+i} & J_2^*(dx_{2n+i}) &= -dx_i & J_3^*(dx_{2n+i}) &= -dx_{6n+i} \\
J_1^*(dx_{3n+i}) &= dx_{5n+i} & J_2^*(dx_{3n+i}) &= dx_{6n+i} & J_3^*(dx_{3n+i}) &= -dx_i \\
J_1^*(dx_{4n+i}) &= -dx_{2n+i} & J_2^*(dx_{4n+i}) &= dx_{n+i} & J_3^*(dx_{4n+i}) &= dx_{7n+i} \\
J_1^*(dx_{5n+i}) &= -dx_{3n+i} & J_2^*(dx_{5n+i}) &= -dx_{7n+i} & J_3^*(dx_{5n+i}) &= dx_{n+i} \\
J_1^*(dx_{6n+i}) &= dx_{7n+i} & J_2^*(dx_{6n+i}) &= -dx_{3n+i} & J_3^*(dx_{6n+i}) &= dx_{2n+i} \\
J_1^*(dx_{7n+i}) &= -dx_{6n+i} & J_2^*(dx_{7n+i}) &= dx_{5n+i} & J_3^*(dx_{7n+i}) &= -dx_{4n+i} \\
J_4^*(dx_i) &= dx_{4n+i} & J_5^*(dx_i) &= dx_{5n+i} & J_6^*(dx_i) &= dx_{6n+i} \\
J_4^*(dx_{n+i}) &= -dx_{2n+i} & J_5^*(dx_{n+i}) &= -dx_{3n+i} & J_6^*(dx_{n+i}) &= -dx_{7n+i} \\
J_4^*(dx_{2n+i}) &= dx_{n+i} & J_5^*(dx_{2n+i}) &= -dx_{7n+i} & J_6^*(dx_{2n+i}) &= -dx_{3n+i} \\
J_4^*(dx_{3n+i}) &= -dx_{7n+i} & J_5^*(dx_{3n+i}) &= dx_{n+i} & J_6^*(dx_{3n+i}) &= dx_{2n+i} \\
J_4^*(dx_{4n+i}) &= -dx_i & J_5^*(dx_{4n+i}) &= dx_{6n+i} & J_6^*(dx_{4n+i}) &= dx_{5n+i} \\
J_4^*(dx_{5n+i}) &= dx_{6n+i} & J_5^*(dx_{5n+i}) &= -dx_i & J_6^*(dx_{5n+i}) &= -dx_{4n+i} \\
J_4^*(dx_{6n+i}) &= -dx_{5n+i} & J_5^*(dx_{6n+i}) &= -dx_{4n+i} & J_6^*(dx_{6n+i}) &= -dx_i \\
J_4^*(dx_{7n+i}) &= dx_{3n+i} & J_5^*(dx_{7n+i}) &= dx_{2n+i} & J_6^*(dx_{7n+i}) &= dx_{n+i}.
\end{aligned} \tag{8}$$

### 3 Hamilton Mechanics

In this section, we obtain Hamilton equations and Hamilton mechanical system for quantum and classical mechanics by means of a canonical local basis  $\{J_1^*, J_2^*, J_3^*, J_4^*, J_5^*, J_6^*\}$  of  $V$  on Clifford Kähler manifold  $(M, V)$ . We saw that the Hamilton equations using basis  $\{J_1^*, J_2^*, J_3^*\}$  of  $V$  on  $(\mathbf{R}^{8n}, V)$  are introduced in [7]. In this study, it is seen that they are the same as the

equations obtained by operators  $J_1^*, J_2^*, J_3^*$  of  $V$  on Clifford Kähler manifold  $(M, V)$ . If we redetermine them, they are respectively:

first:

$$\begin{aligned}\frac{dx_i}{dt} &= -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \quad \frac{dx_{2n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{4n+i}}, \quad \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{5n+i}}, \\ \frac{dx_{4n+i}}{dt} &= \frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \quad \frac{dx_{5n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \quad \frac{dx_{6n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \quad \frac{dx_{7n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{6n+i}}.\end{aligned}$$

second:

$$\begin{aligned}\frac{dx_i}{dt} &= -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{4n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \quad \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{6n+i}}, \\ \frac{dx_{4n+i}}{dt} &= -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \quad \frac{dx_{5n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \quad \frac{dx_{6n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \quad \frac{dx_{7n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{5n+i}}.\end{aligned}$$

third:

$$\begin{aligned}\frac{dx_i}{dt} &= -\frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{5n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{6n+i}}, \quad \frac{dx_{3n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \\ \frac{dx_{4n+i}}{dt} &= -\frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \quad \frac{dx_{5n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \quad \frac{dx_{6n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \quad \frac{dx_{7n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{4n+i}}.\end{aligned}$$

Fourth, let  $(M, V)$  be a Clifford Kähler manifold. Suppose that a component of almost Clifford structure  $V^*$ , a Liouville form and a 1-form on Clifford Kähler manifold  $(M, V)$  are given by  $J_4^*$ ,  $\lambda_{J_4^*}$  and  $\omega_{J_4^*}$ , respectively.

Putting

$$\begin{aligned}\omega_{J_4^*} &= \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} + x_{2n+i} dx_{2n+i} + x_{3n+i} dx_{3n+i} \\ &\quad + x_{4n+i} dx_{4n+i} + x_{5n+i} dx_{5n+i} + x_{6n+i} dx_{6n+i} + x_{7n+i} dx_{7n+i}),\end{aligned}$$

we have

$$\begin{aligned}\lambda_{J_4^*} &= J_4^*(\omega_{J_4^*}) = \frac{1}{2}(x_i dx_{4n+i} - x_{n+i} dx_{2n+i} + x_{2n+i} dx_{n+i} - x_{3n+i} dx_{7n+i} \\ &\quad - x_{4n+i} dx_i + x_{5n+i} dx_{6n+i} - x_{6n+i} dx_{5n+i} + x_{7n+i} dx_{3n+i}).\end{aligned}$$

It is known that if  $\Phi_{J_4^*}$  is a closed Kähler form on Clifford Kähler manifold  $(M, V)$ , then  $\Phi_{J_4^*}$  is also a symplectic structure on Clifford Kähler manifold  $(M, V)$ .

Take into consideration that Hamilton vector field  $X$  associated with Hamilton energy  $\mathbf{H}$  is given by

$$\begin{aligned} X = & X^i \frac{\partial}{\partial x_i} + X^{n+i} \frac{\partial}{\partial x_{n+i}} + X^{2n+i} \frac{\partial}{\partial x_{2n+i}} + X^{3n+i} \frac{\partial}{\partial x_{3n+i}} \\ & + X^{4n+i} \frac{\partial}{\partial x_{4n+i}} + X^{5n+i} \frac{\partial}{\partial x_{5n+i}} + X^{6n+i} \frac{\partial}{\partial x_{6n+i}} + X^{7n+i} \frac{\partial}{\partial x_{7n+i}}. \end{aligned} \quad (9)$$

Then

$$\Phi_{J_4^*} = -d\lambda_{J_4^*} = dx_{n+i} \wedge dx_{2n+i} + dx_{3n+i} \wedge dx_{7n+i} + dx_{4n+i} \wedge dx_i + dx_{6n+i} \wedge dx_{5n+i} \quad (10)$$

and

$$\begin{aligned} i_X \Phi_{J_4^*} = \Phi_{J_4^*}(X) = & X^{n+i} dx_{2n+i} - X^{2n+i} dx_{n+i} + X^{3n+i} dx_{7n+i} - X^{7n+i} dx_{3n+i} \\ & + X^{4n+i} dx_i - X^i dx_{4n+i} + X^{6n+i} dx_{5n+i} - X^{5n+i} dx_{6n+i}. \end{aligned} \quad (11)$$

Furthermore, the differential of Hamilton energy is obtained as follows:

$$\begin{aligned} d\mathbf{H} = & \frac{\partial \mathbf{H}}{\partial x_i} dx_i + \frac{\partial \mathbf{H}}{\partial x_{n+i}} dx_{n+i} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}} dx_{2n+i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} dx_{3n+i} \\ & + \frac{\partial \mathbf{H}}{\partial x_{4n+i}} dx_{4n+i} + \frac{\partial \mathbf{H}}{\partial x_{5n+i}} dx_{5n+i} + \frac{\partial \mathbf{H}}{\partial x_{6n+i}} dx_{6n+i} + \frac{\partial \mathbf{H}}{\partial x_{7n+i}} dx_{7n+i}. \end{aligned} \quad (12)$$

According to **Eq.**(1), if equaled **Eq.** (11) and **Eq.** (12), the Hamilton vector field is calculated as follows:

$$\begin{aligned} X = & -\frac{\partial \mathbf{H}}{\partial x_{4n+i}} \frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{2n+i}} + \frac{\partial \mathbf{H}}{\partial x_{7n+i}} \frac{\partial}{\partial x_{3n+i}} \\ & + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{4n+i}} - \frac{\partial \mathbf{H}}{\partial x_{6n+i}} \frac{\partial}{\partial x_{5n+i}} + \frac{\partial \mathbf{H}}{\partial x_{5n+i}} \frac{\partial}{\partial x_{6n+i}} - \frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_{7n+i}} \end{aligned} \quad (13)$$

Assume that a curve

$$\alpha : \mathbf{R} \rightarrow M \quad (14)$$

be an integral curve of the Hamilton vector field  $X$ , i.e.,

$$X(\alpha(t)) = \dot{\alpha}, \quad t \in \mathbf{R}. \quad (15)$$



In the local coordinates, it is found that

$$\alpha(t) = (x_i, x_{n+i}, x_{2n+i}, x_{3n+i}, x_{4n+i}, x_{5n+i}, x_{6n+i}, x_{7n+i}) \quad (16)$$

and

$$\begin{aligned} \dot{\alpha}(t) = & \frac{dx_i}{dt} \frac{\partial}{\partial x_i} + \frac{dx_{n+i}}{dt} \frac{\partial}{\partial x_{n+i}} + \frac{dx_{2n+i}}{dt} \frac{\partial}{\partial x_{2n+i}} + \frac{dx_{3n+i}}{dt} \frac{\partial}{\partial x_{3n+i}} \\ & + \frac{dx_{4n+i}}{dt} \frac{\partial}{\partial x_{4n+i}} + \frac{dx_{5n+i}}{dt} \frac{\partial}{\partial x_{5n+i}} + \frac{dx_{6n+i}}{dt} \frac{\partial}{\partial x_{6n+i}} + \frac{dx_{7n+i}}{dt} \frac{\partial}{\partial x_{7n+i}}. \end{aligned} \quad (17)$$

Thinking out **Eq.** (15), if equaled **Eq.** (13) and **Eq.** (17), it follows

$$\begin{aligned} \frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{4n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \quad \frac{dx_{2n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \quad \frac{dx_{3n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \\ \frac{dx_{4n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \quad \frac{dx_{5n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{6n+i}}, \quad \frac{dx_{6n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{5n+i}}, \quad \frac{dx_{7n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}}. \end{aligned} \quad (18)$$

Hence, the equations obtained in **Eq.** (18) are shown to be *Hamilton equations* with respect to component  $J_4^*$  of almost Clifford structure  $V^*$  on Clifford Kähler manifold  $(M, V)$ , and then the triple  $(M, \Phi_{J_4^*}, X)$  is said to be a *Hamilton mechanical system* on Clifford Kähler manifold  $(M, V)$ .

Fifth, let  $(M, V)$  be a Clifford Kähler manifold. Assume that an element of almost Clifford structure  $V^*$ , a Liouville form and a 1-form on Clifford Kähler manifold  $(M, V)$  are determined by  $J_5^*$ ,  $\lambda_{J_5^*}$  and  $\omega_{J_5^*}$ , respectively.

Setting

$$\begin{aligned} \omega_{J_5^*} = & \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} + x_{2n+i} dx_{2n+i} + x_{3n+i} dx_{3n+i} \\ & + x_{4n+i} dx_{4n+i} + x_{5n+i} dx_{5n+i} + x_{6n+i} dx_{6n+i} + x_{7n+i} dx_{7n+i}), \end{aligned}$$

we have

$$\begin{aligned} \lambda_{J_5^*} = & J_5^*(\omega_{J_5^*}) = \frac{1}{2}(x_i dx_{5n+i} - x_{n+i} dx_{3n+i} - x_{2n+i} dx_{7n+i} + x_{3n+i} dx_{n+i} \\ & + x_{4n+i} dx_{6n+i} - x_{5n+i} dx_i - x_{6n+i} dx_{4n+i} + x_{7n+i} dx_{2n+i}). \end{aligned}$$

Assume that  $X$  is a Hamilton vector field related to Hamilton energy  $\mathbf{H}$  and given by **Eq.** (9).

Take into consideration

$$\Phi_{J_5^*} = -d\lambda_{J_5^*} = dx_{n+i} \wedge dx_{3n+i} + dx_{2n+i} \wedge dx_{7n+i} + dx_{5n+i} \wedge dx_i + dx_{6n+i} \wedge dx_{4n+i}, \quad (19)$$

then we find

$$\begin{aligned} i_X \Phi_{J_5^*} = \Phi_{J_5^*}(X) = & X^{n+i} dx_{3n+i} - X^{3n+i} dx_{n+i} + X^{2n+i} dx_{7n+i} - X^{7n+i} dx_{2n+i} \\ & + X^{5n+i} dx_i - X^i dx_{5n+i} + X^{6n+i} dx_{4n+i} - X^{4n+i} dx_{6n+i}. \end{aligned} \quad (20)$$

According to **Eq.**(1), if we equal **Eq.** (12) and **Eq.** (20), it follows

$$\begin{aligned} X = & -\frac{\partial \mathbf{H}}{\partial x_{5n+i}} \frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_{7n+i}} \frac{\partial}{\partial x_{2n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{3n+i}} \\ & - \frac{\partial \mathbf{H}}{\partial x_{6n+i}} \frac{\partial}{\partial x_{4n+i}} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{5n+i}} + \frac{\partial \mathbf{H}}{\partial x_{4n+i}} \frac{\partial}{\partial x_{6n+i}} - \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{7n+i}} \end{aligned} \quad (21)$$

Taking **Eq.** (15), **Eqs.** (17) and (21) are equal, we obtain equations

$$\begin{aligned} \frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{5n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \quad \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \\ \frac{dx_{4n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{6n+i}}, \quad \frac{dx_{5n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \quad \frac{dx_{6n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{4n+i}}, \quad \frac{dx_{7n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}. \end{aligned} \quad (22)$$

In the end, the equations found in **Eq.** (22) are seen to be *Hamilton equations* with respect to component  $J_5^*$  of almost Clifford structure  $V^*$  on Clifford Kähler manifold  $(M, V)$ , and then the triple  $(M, \Phi_{J_5^*}, X)$  is named a *Hamilton mechanical system* on Clifford Kähler manifold  $(M, V)$ .

Sixth, let  $(M, V)$  be a Clifford Kähler manifold. By  $J_6^*$ ,  $\lambda_{J_6^*}$  and  $\omega_{J_6^*}$ , we denote a component of almost Clifford structure  $V^*$ , a Liouville form and a 1-form on Clifford Kähler manifold  $(M, V)$ , respectively.

Let  $\omega_{J_6^*}$  be determined by

$$\begin{aligned}\omega_{J_6^*} = & \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} + x_{2n+i} dx_{2n+i} + x_{3n+i} dx_{3n+i} \\ & + x_{4n+i} dx_{4n+i} + x_{5n+i} dx_{5n+i} + x_{6n+i} dx_{6n+i} + x_{7n+i} dx_{7n+i})\end{aligned}$$

Then it yields

$$\begin{aligned}\lambda_{J_6^*} = & J_6^*(\omega_{J_6^*}) = \frac{1}{2}(x_i dx_{6n+i} - x_{n+i} dx_{7n+i} - x_{2n+i} dx_{3n+i} + x_{3n+i} dx_{2n+i} \\ & + x_{4n+i} dx_{5n+i} - x_{5n+i} dx_{4n+i} - x_{6n+i} dx_i + x_{7n+i} dx_{n+i}).\end{aligned}$$

It is known that if  $\Phi_{J_6^*}$  is a closed Kähler form on Clifford Kähler manifold  $(M, V)$ , then  $\Phi_{J_6^*}$  is also a symplectic structure on Clifford Kähler manifold  $(M, V)$ .

Take  $X$ . It is Hamilton vector field connected with Hamilton energy  $\mathbf{H}$  and given by **Eq.** (9).

Considering

$$\Phi_{J_6^*} = -d\lambda_{J_6^*} = dx_{n+i} \wedge dx_{7n+i} + dx_{2n+i} \wedge dx_{3n+i} + dx_{5n+i} \wedge dx_{4n+i} + dx_{6n+i} \wedge dx_i, \quad (23)$$

we calculate

$$\begin{aligned}i_X \Phi_{J_6^*} = \Phi_{J_6^*}(X) = & X^{n+i} dx_{7n+i} - X^{7n+i} dx_{n+i} + X^{2n+i} dx_{3n+i} - X^{3n+i} dx_{2n+i} \\ & + X^{5n+i} dx_{4n+i} - X^{4n+i} dx_{5n+i} + X^{6n+i} dx_i - X^i dx_{6n+i}.\end{aligned} \quad (24)$$

According to **Eq.**(1), **Eqs.** (12) and (24) are equaled, Hamilton vector field is found as follows:

$$\begin{aligned}X = & -\frac{\partial \mathbf{H}}{\partial x_{6n+i}} \frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{7n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_{2n+i}} - \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{3n+i}} \\ & - \frac{\partial \mathbf{H}}{\partial x_{5n+i}} \frac{\partial}{\partial x_{4n+i}} + \frac{\partial \mathbf{H}}{\partial x_{4n+i}} \frac{\partial}{\partial x_{5n+i}} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{6n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{7n+i}}.\end{aligned} \quad (25)$$

Considering **Eq.** (15), we equal **Eq.** (17) and **Eq.** (25), it holds

$$\begin{aligned} \frac{dx_i}{dt} &= -\frac{\partial \mathbf{H}}{\partial x_{6n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \quad \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \\ \frac{dx_{4n+i}}{dt} &= -\frac{\partial \mathbf{H}}{\partial x_{5n+i}}, \quad \frac{dx_{5n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{4n+i}}, \quad \frac{dx_{6n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \quad \frac{dx_{7n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}. \end{aligned} \quad (26)$$

Finally, the equations calculated in **Eq.** (26) are called to be *Hamilton equations* with respect to component  $J_6^*$  of almost Clifford structure  $V^*$  on Clifford Kähler manifold  $(M, V)$ , and then the triple  $(M, \Phi_{J_6^*}, X)$  is said to be a *Hamilton mechanical system* on Clifford Kähler manifold  $(M, V)$ .

## 4 Conclusion

Hamilton Formalisms has intrinsically been described with taking into account the basis  $\{J_1^*, J_2^*, J_3^*, J_4^*, J_5^*,$  of almost Clifford structure  $V^*$  on Clifford Kähler manifold  $(M, V)$ .

Hamilton models arise to be a very important tool since they present a simple method to describe the model for dynamical systems. In solving problems in classical mechanics, the rotational mechanical system will then be easily usable model.

Since a new model for dynamic systems on subspaces and spaces is needed, equations (18), (22) and (26) are only considered to be a first step to realize how Clifford geometry has been used in understanding, modeling and solving problems in different physical fields.

For further research, the Hamilton vector fields and equations obtained here are advised to deal with problems in applicable fields of quantum and classical mechanics of physics.

## References

- [1] M. De Leon, P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, North-Holland Mathematics Studies, vol.152, Elsevier, Amsterdam, 1989.
- [2] M. Tekkoyun, On Para-Euler–Lagrange and Para-Hamiltonian Equations , Phys. Lett. A, Vol. 340, Issues 1-4, 2005, pp. 7-11
- [3] D. Stahlke, Quaternions in Classical Mechanics, Phys 621.  
<http://www.stahlke.org/dan/phys-papers/quaternion-paper.pdf>
- [4] K. Yano, M. Kon, Structures on Manifolds, Series in Pure Mathematics-Volume 3, World Scientific Publishing Co. Pte. Ltd., Singore, 1984.
- [5] I. Burdujan, Clifford Kähler Manifolds, Balkan Journal of Geometry and its Applications, Vol.13, No:2, 2008, pp.12-23
- [6] M. Tekkoyun, Lagrangian Mechanics on the Standard Cliffordian Kähler Manifolds, arXiv:0902.3724v1
- [7] M. Tekkoyun, Hamiltonian Mechanic Systems on the Standard Cliffordian Kähler Manifolds, arXiv:0902.3867v1